

# The Celestial Sphere

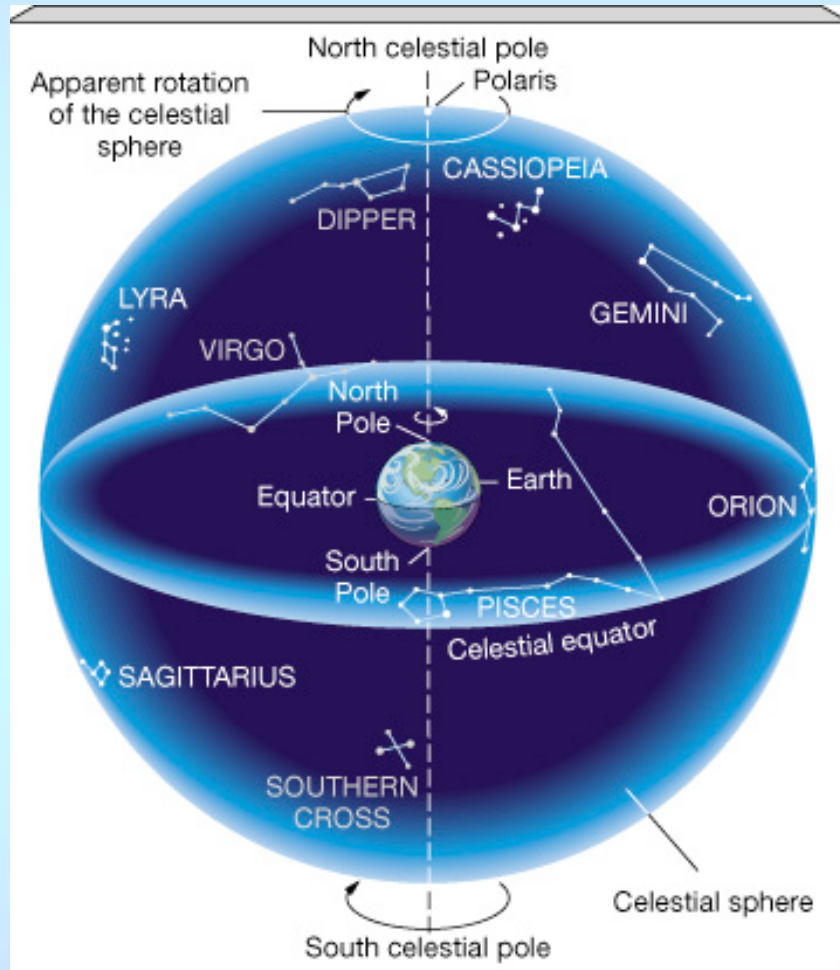
## Useful References:

- Smart, “Text-Book on Spherical Astronomy” (or similar)
- “Astronomical Almanac” and “Astronomical Almanac’s Explanatory Supplement” (always definitive)
- Lang, “Astrophysical Formulae” (for quick reference)
- Allen “Astrophysical Quantities” (for quick reference)
- Karttunen, “Fundamental Astronomy” (e-book version accessible from Penn State at  
<http://www.springerlink.com/content/j5658r/>

# Numbers to Keep in Mind

- $4 \pi (180 / \pi)^2 = 41,253 \text{ deg}^2$  on the sky
- $\sim 23.5^\circ$  = obliquity of the ecliptic
- $17^{\text{h}} 45^{\text{m}}, -29^\circ$  = coordinates of Galactic Center
- $12^{\text{h}} 51^{\text{m}}, +27^\circ$  = coordinates of North Galactic Pole
- $18^{\text{h}}, +66^\circ 33'$  = coordinates of North Ecliptic Pole

# Spherical Astronomy



Geocentrically speaking, the Earth sits inside a celestial sphere containing fixed stars. We are therefore driven towards equations based on spherical coordinates.

# Rules for Spherical Astronomy

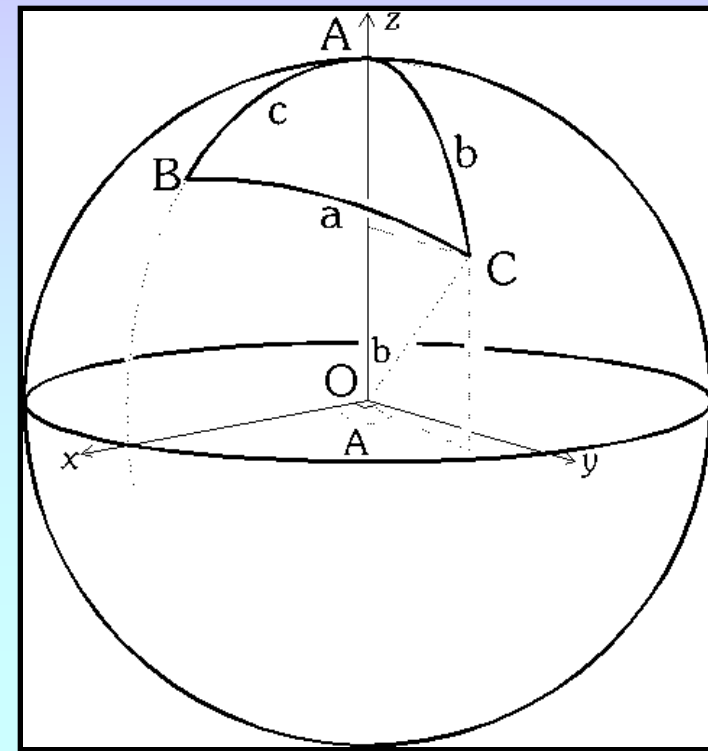
- The shortest distance between two points on a sphere is a great circle.
- The length of a (great circle) arc is proportional to the angle created by the two radial vectors defining the points.
- The great-circle arc length between two points on a sphere is given by

$$\cos a = (\cos b \cos c) + (\sin b \sin c \cos A)$$

(This is the fundamental equation of spherical trigonometry.)

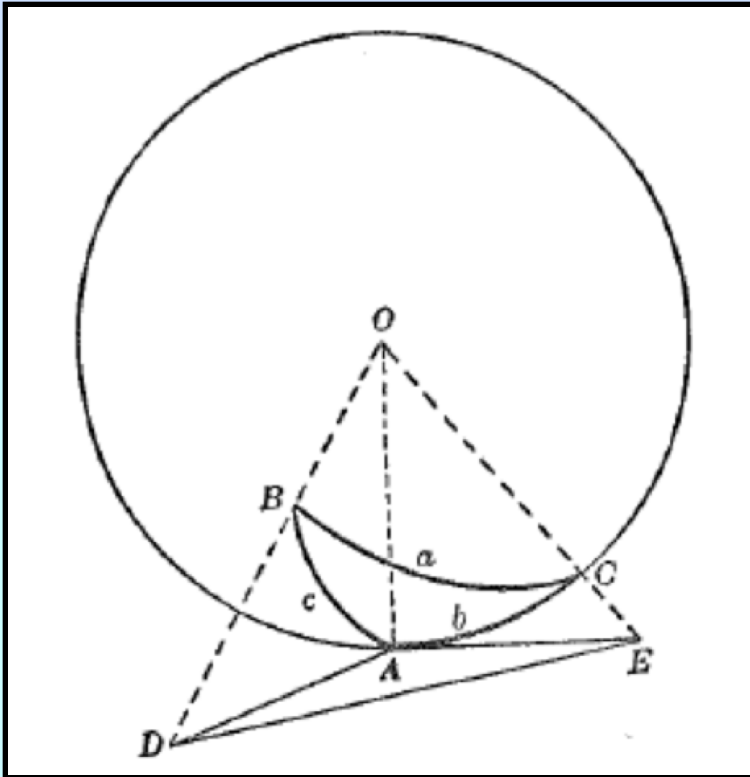
- Two other spherical triangle relations which can be derived from the fundamental equation are

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} \quad \text{and} \quad \sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$$





# Proof of Fundamental Equation



- O is at the center of a unit sphere, AD and AE are tangent to the sphere at A, and angles  $BAC = \hat{A}$  and  $DOE = \hat{O}$
- OAD and OAE are right angles, so  
 $AD = \tan c$     $OD = \sec c$   
 $AE = \tan b$     $OE = \sec b$
- The law of cosines on  $\triangle DAE$  gives  
 $DE^2 = AD^2 + AE^2 - 2 AD AE \cos \hat{A}$   
 $= \tan^2 c + \tan^2 b - 2 \tan c \tan b \cos \hat{A}$

- Similarly, through  $\triangle DOE$ ,  $DE^2 = OD^2 + OE^2 - 2 OD OE \cos \hat{O}$
- Since for  $\triangle BOC$ ,  $\hat{O} = a$ ,  $DE^2 = \sec^2 c + \sec^2 b - 2 \sec b \sec c \cos a$
- Setting the two equations for  $DE^2$  equal, then yields  
 $\sec^2 c + \sec^2 b - 2 \sec b \sec c \cos a = \tan^2 c + \tan^2 b - 2 \tan b \tan c \cos \hat{A}$
- Since  $\sec^2 = 1 + \tan^2$ ,  $\cos a = (\cos b \cos c) + (\sin b \sin c \cos \hat{A})$

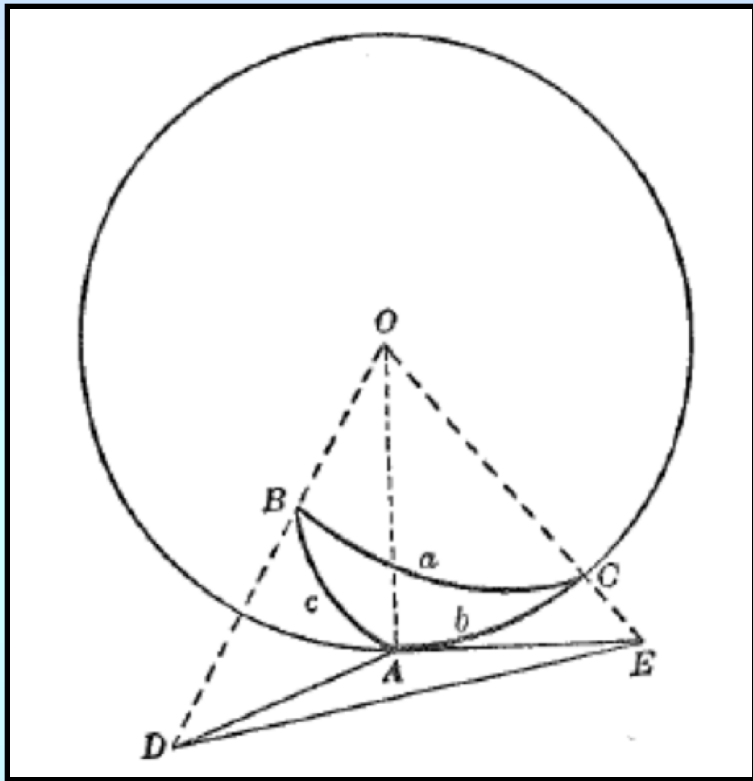
# More on the Fundamental Equation

One note on the fundamental equation:  
$$\cos a = (\cos b \cos c) + (\sin b \sin c \cos \hat{A})$$

As you can see, the arcs  $b$  and  $c$  are measured from the poles. In astronomy, however, one is usually dealing with the complement of these numbers, such as a latitude. Consequently, most of the time you are using

$$\cos \theta = \sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos (\alpha_1 - \alpha_2)$$

where  $\delta$  represents a latitude-like coordinate, and  $\alpha$  represents a longitude-like coordinate.



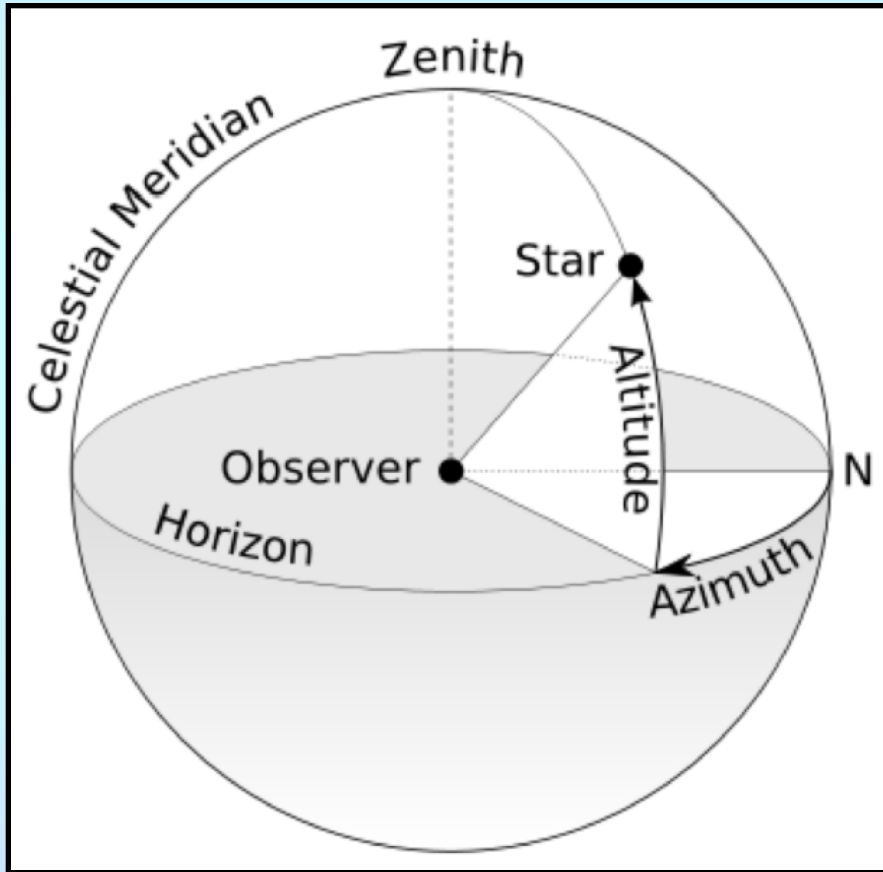
# Coordinate Systems

There are at least 5 types of spherical coordinate systems that are commonly used in astronomy

- 1) **Horizon Coordinates** (altitude-azimuth): defined by the place and time of observation ( $a, A$ )
- 2) **Equatorial Coordinates** (right ascension – declination): defined by the Earth's rotation axis ( $\alpha, \delta$ )
- 3) **Ecliptic Coordinates**: defined by Earth's ecliptic plane ( $\lambda, \beta$ )
- 4) **Galactic Coordinates**: defined by the plane of the Milky Way ( $\ell, b$ )
- 5) **Supergalactic Coordinates**: defined by the large scale structure of the local universe ( $L, B$ )

# The Horizon System

The simplest coordinate system used is the horizon system. Positions are defined via the altitude,  $a$  (or, its complement, zenith angle,  $z$ ), and the azimuth angle,  $A$  (north-through-east).

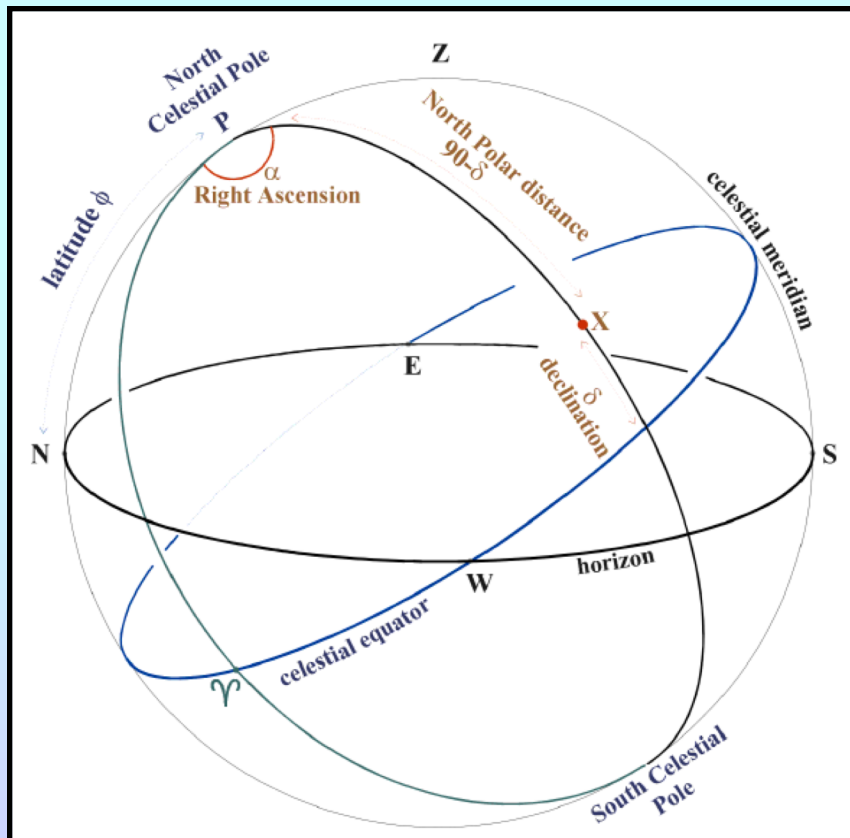


The **meridian** is the great circle that goes through the zenith, the nadir, and the celestial poles. (It divides east and west.) **Hour angle** ( $H$ ) is the time until (or after) an object transits the meridian.

Because the Earth is rotating, a star's horizon coordinates will depend on the place and time of the observation.

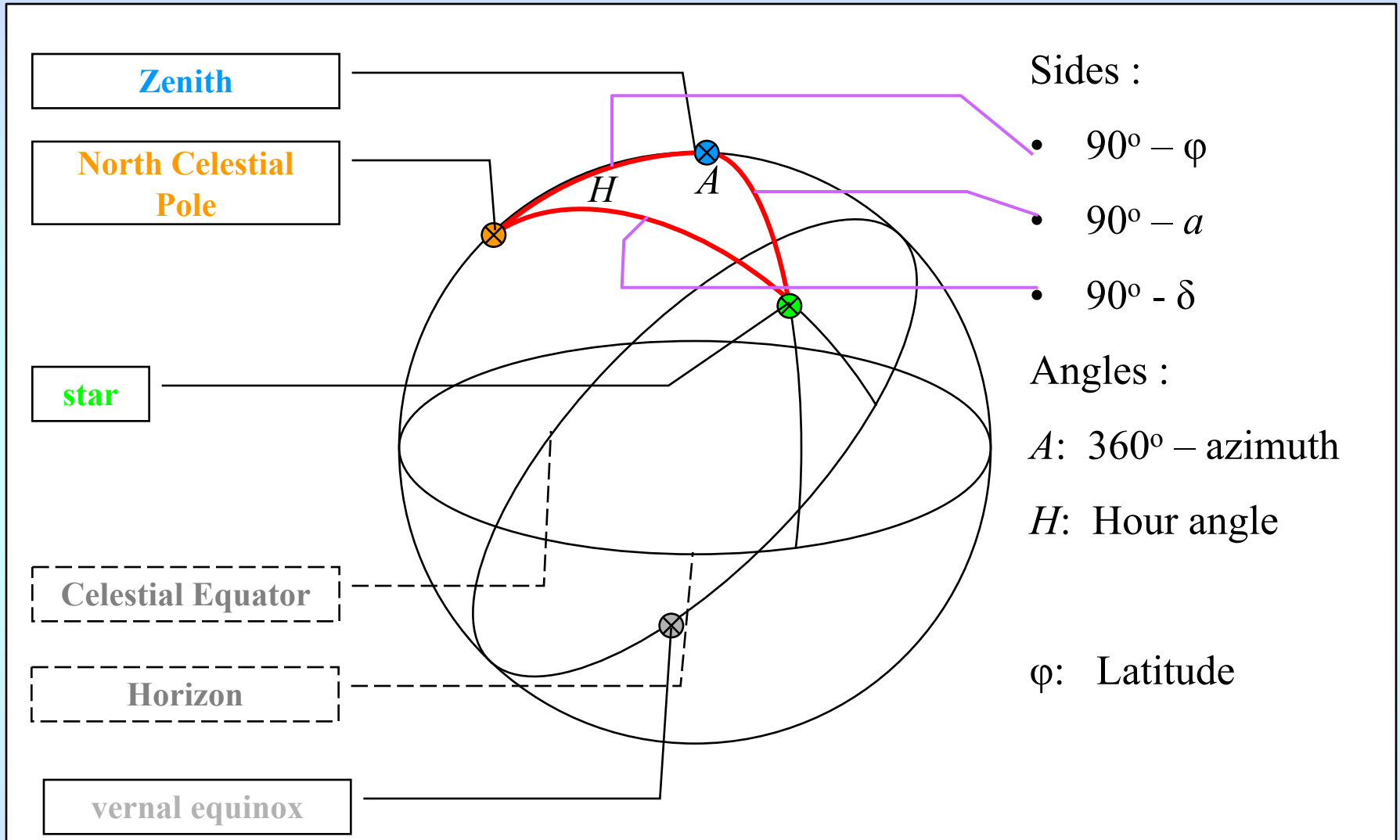
# The Equatorial System

The most common coordinate system used is the equatorial system. An object's **declination** ( $\delta$ ) is equivalent to its latitude:  $+90^\circ$  is over the north pole,  $-90^\circ$  is over the south pole, and  $0^\circ$  is over the equator. **Right ascension** ( $\alpha$ ) is equivalent to longitude, but is *usually* measured over 24 hours, rather than  $360^\circ$ .



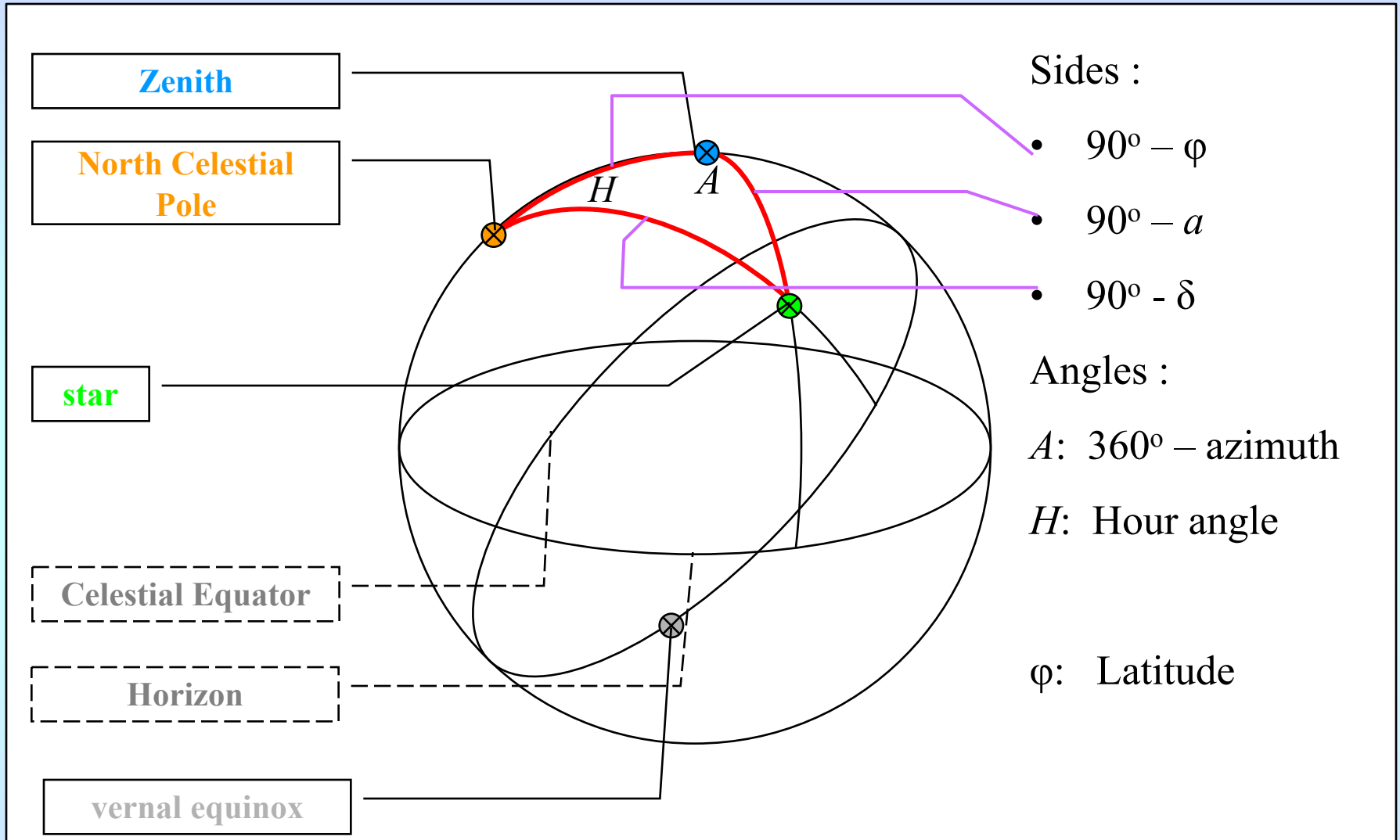
The zero point of right ascension is the location of the Sun on the vernal equinox, i.e., the intersection of the ecliptic plane with the celestial equator, on the side where the Sun is ascending from the south to the north.

# Horizon – Equatorial Conversion



To translate horizon coordinates  $(a, A)$  to hour angle and declination  $(H, \delta)$  examine the red highlighted triangle.

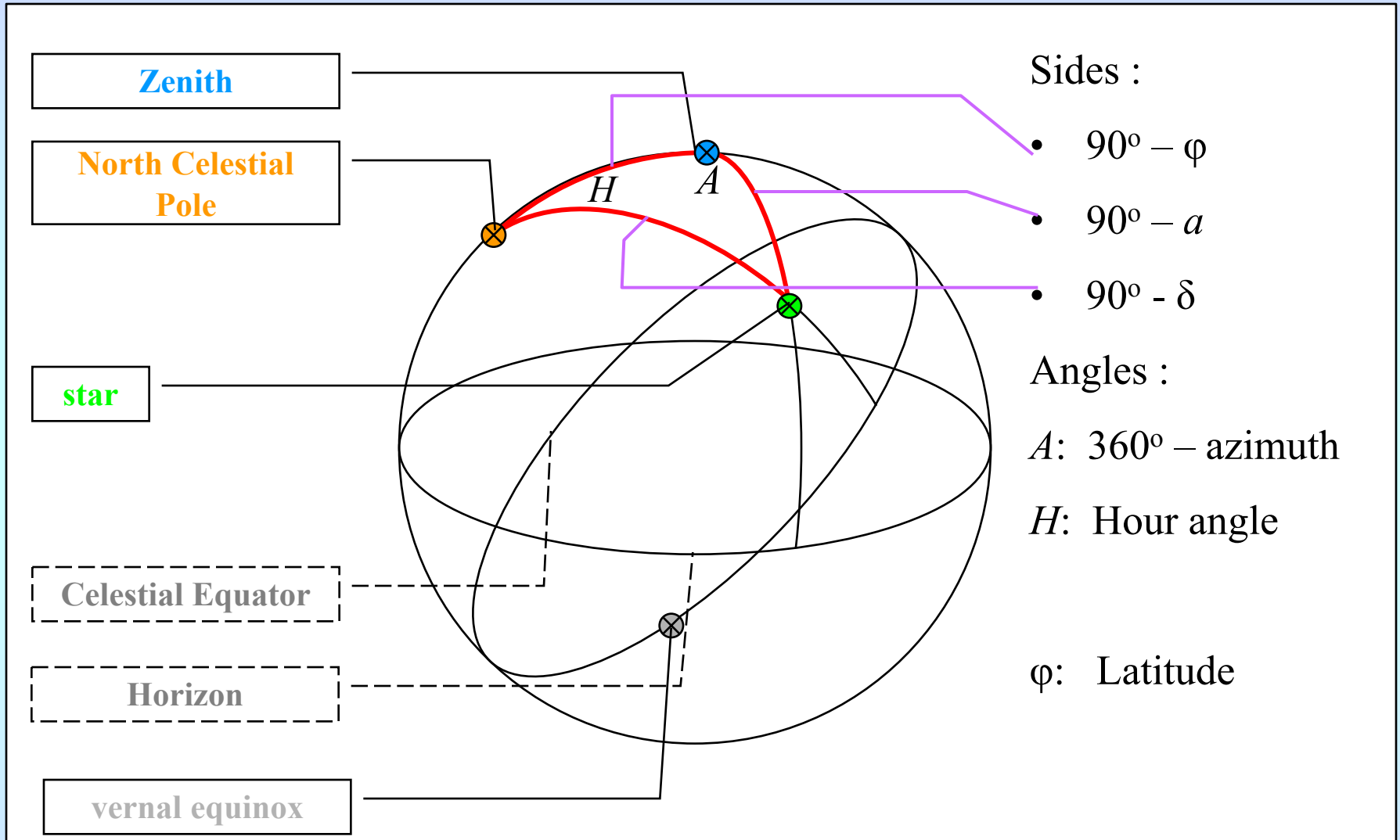
# Horizon – Equatorial Conversion



$$\cos(90-\delta) = \cos(90-\varphi) \cos(90-a) + \sin(90-\varphi) \sin(90-a) \cos A$$

$$\text{or} \quad \sin \delta = \sin \varphi \sin a + \cos \varphi \cos a \cos A$$

# Horizon – Equatorial Conversion



$$\cos(90-a) = \cos(90-\delta) \cos(90-\phi) + \sin(90-\delta) \sin(90-\phi) \cos H$$

$$\text{or } \sin a = \sin \delta \sin \phi + \cos \delta \cos \phi \cos H$$



# Horizon – Equatorial Conversion Summary

With  $H = \text{LST} - \alpha$  and  $\varphi = \text{observer's latitude}$ , the conversions are

$$\cos a \sin A = -\cos \delta \sin H$$

$$\cos a \cos A = \sin \delta \cos \varphi - \cos \delta \cos H \sin \varphi$$

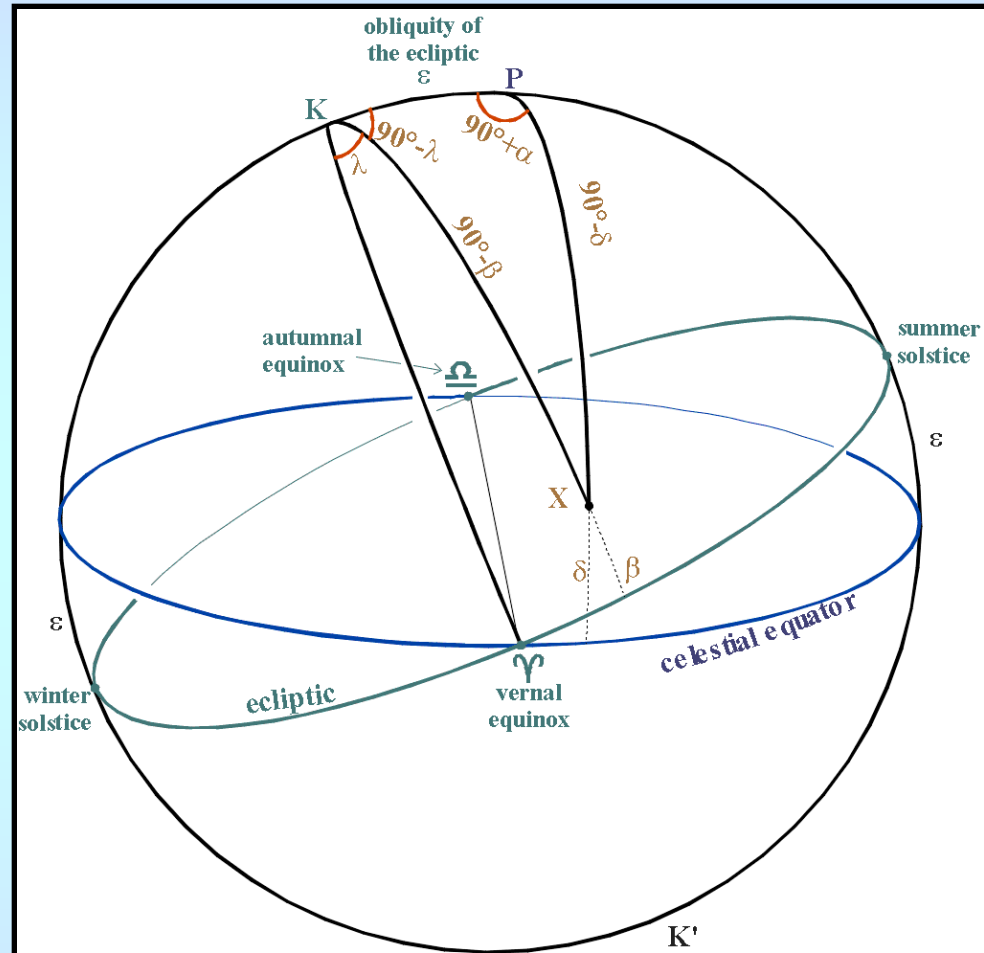
$$\sin a = \sin \delta \sin \varphi + \cos \delta \cos H \cos \varphi$$

$$\cos \delta \cos H = \sin a \cos \varphi - \cos a \cos A \sin \varphi$$

$$\sin \delta = \sin a \sin \varphi + \cos a \cos A \cos \varphi$$

# Ecliptic Coordinates

**Ecliptic coordinates** ( $\lambda$ ,  $\beta$ ) are defined via the plane of the Earth's orbit about the Sun. In this system, the ecliptic pole ( $\beta=90^\circ$ ) is defined as the direction perpendicular to the Earth's orbital plane in the northern part of the sky. Since the Earth's obliquity (tilt) is about  $\varepsilon \approx 23.5^\circ$ , the direction of this pole is close to  $\delta = 66.5^\circ$  (actually,  $\delta = 66^\circ 33' 38.55''$ ). The zero point of longitude ( $\lambda$ ) is the same as that used for equatorial positions: the direction of the vernal equinox.



Unlike right ascension, ecliptic longitude is always measured in degrees.

# Equatorial - Ecliptic Transformations

Because both systems use the same longitude zero point, transformations between the two systems are easy: just use  $\Delta P X K$ . For example:

$$\cos(90-\beta) = \cos \varepsilon \cos (90-\delta) + \sin \varepsilon \sin(90-\delta) \cos(90+\alpha)$$

gives

$$\sin \beta = \cos \varepsilon \sin \delta - \sin \varepsilon \cos \delta \sin \alpha$$

Similarly,

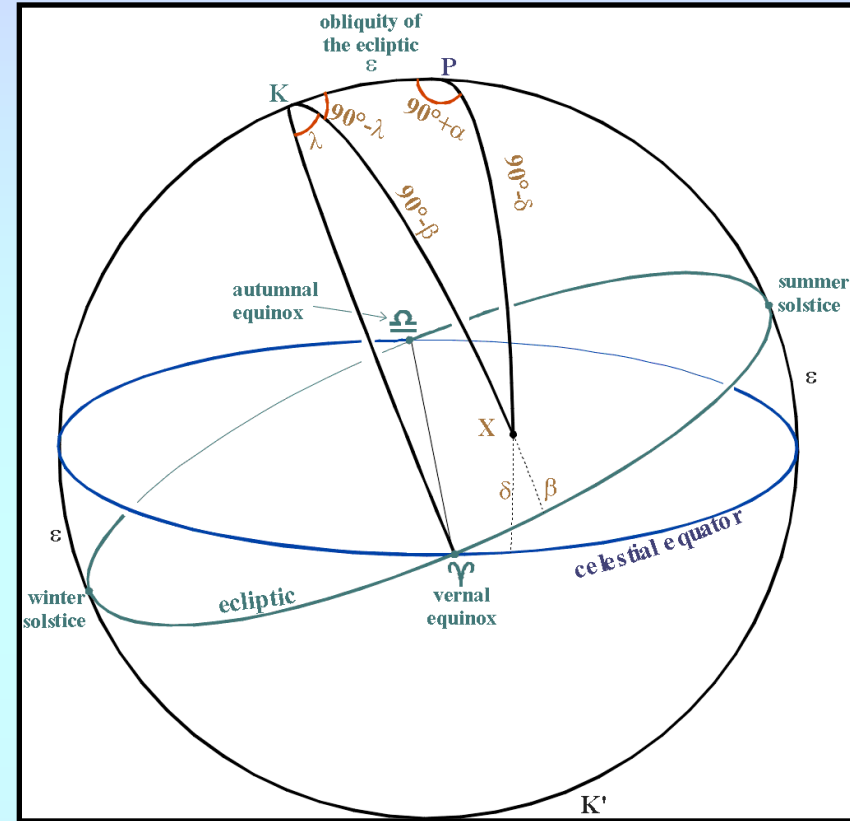
$$\cos \delta \cos \alpha = \cos \beta \sin \lambda$$

$$\cos \delta \sin \alpha = \cos \beta \sin \lambda \cos \varepsilon - \sin \beta \sin \varepsilon$$

$$\sin \delta = \cos \beta \sin \lambda \sin \varepsilon + \sin \beta \cos \varepsilon$$

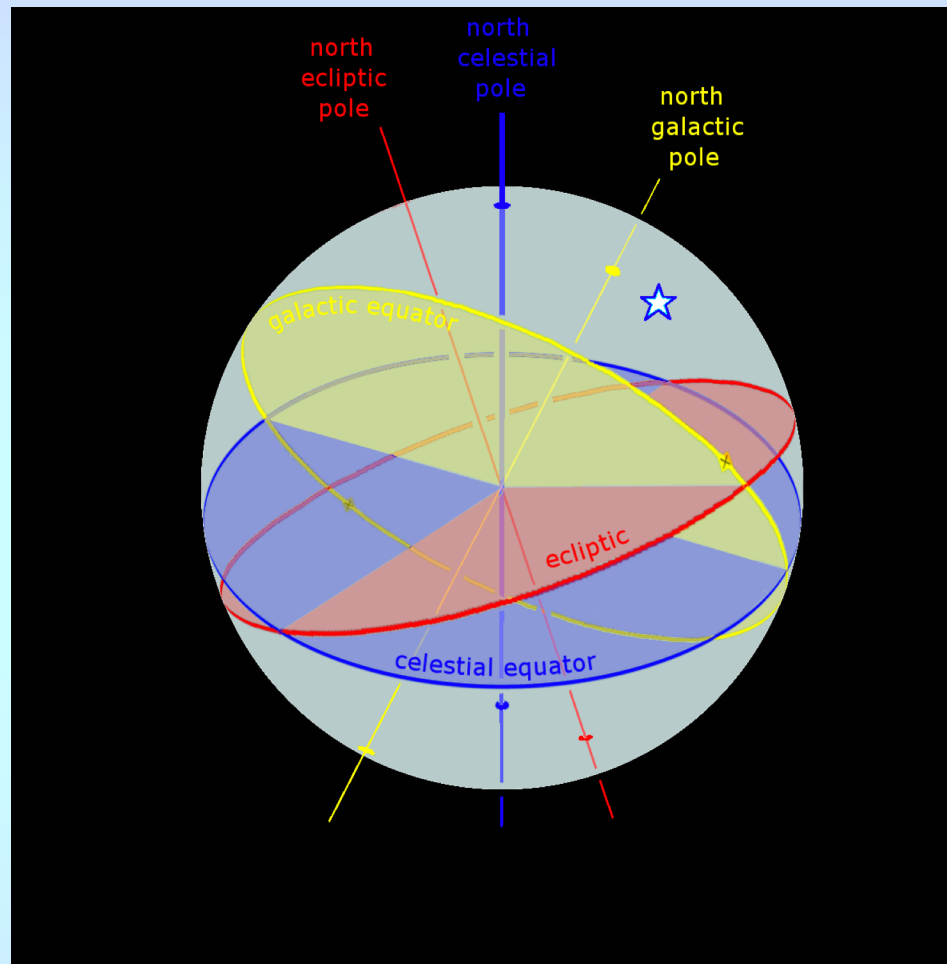
$$\cos \beta \sin \lambda = \cos \delta \sin \alpha \cos \varepsilon + \sin \delta \sin \varepsilon$$

$$\sin \beta = \sin \delta \cos \varepsilon - \cos \delta \sin \alpha \sin \varepsilon$$



# Galactic Coordinates

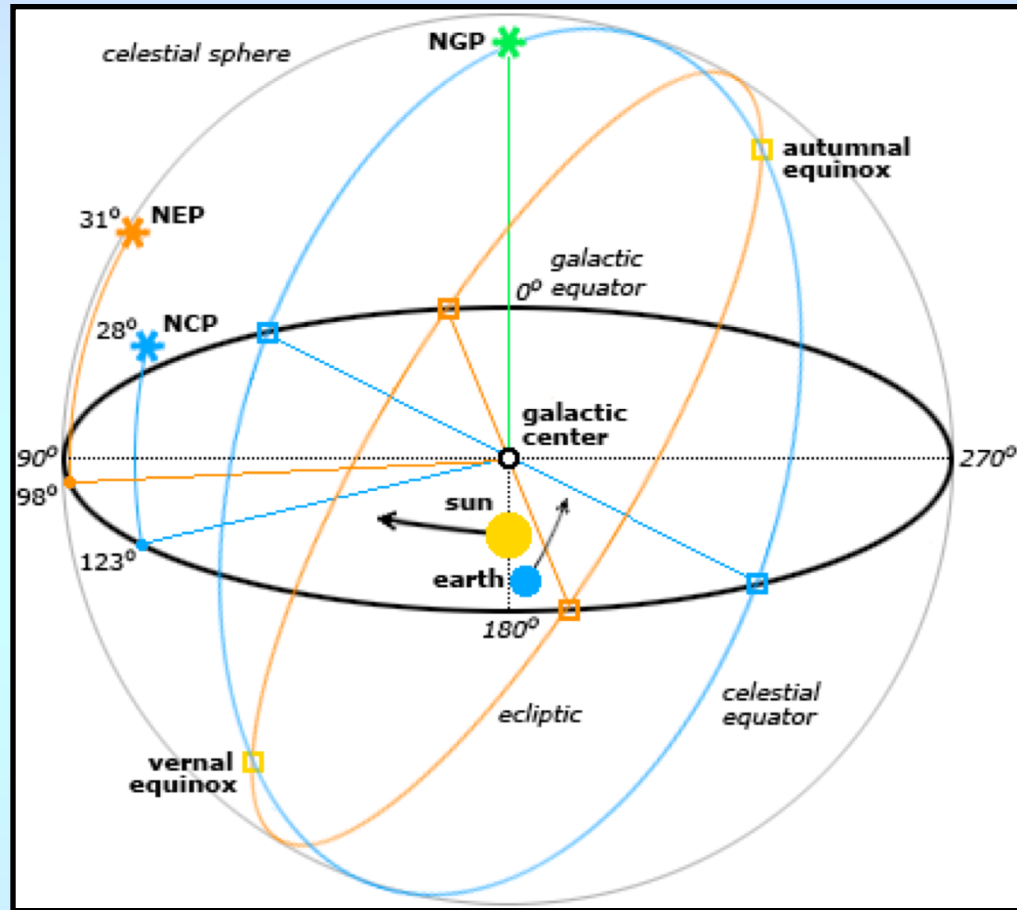
For many problems, **Galactic Coordinates** ( $\ell$ ,  $b$ ) are the most natural. Until 1958, the north galactic pole was defined to be at  $12^{\text{h}}40^{\text{m}}, +28^{\circ}$  (B1900), with  $\ell=0^{\circ}$  being the vernal equinox. (This is the  $\ell^{\text{I}}$ ,  $b^{\text{I}}$  system.) In 1958, however, the IAU moved the Galactic pole to  $12^{\text{h}}49^{\text{m}}, +27^{\circ} 24'$  (B1950) and decoupled longitude from the vernal equinox. Now, the intersection between the Galactic and equatorial planes at the ascending node is defined to be  $18^{\text{h}}49^{\text{m}}$  at  $\ell=33^{\circ}$ .



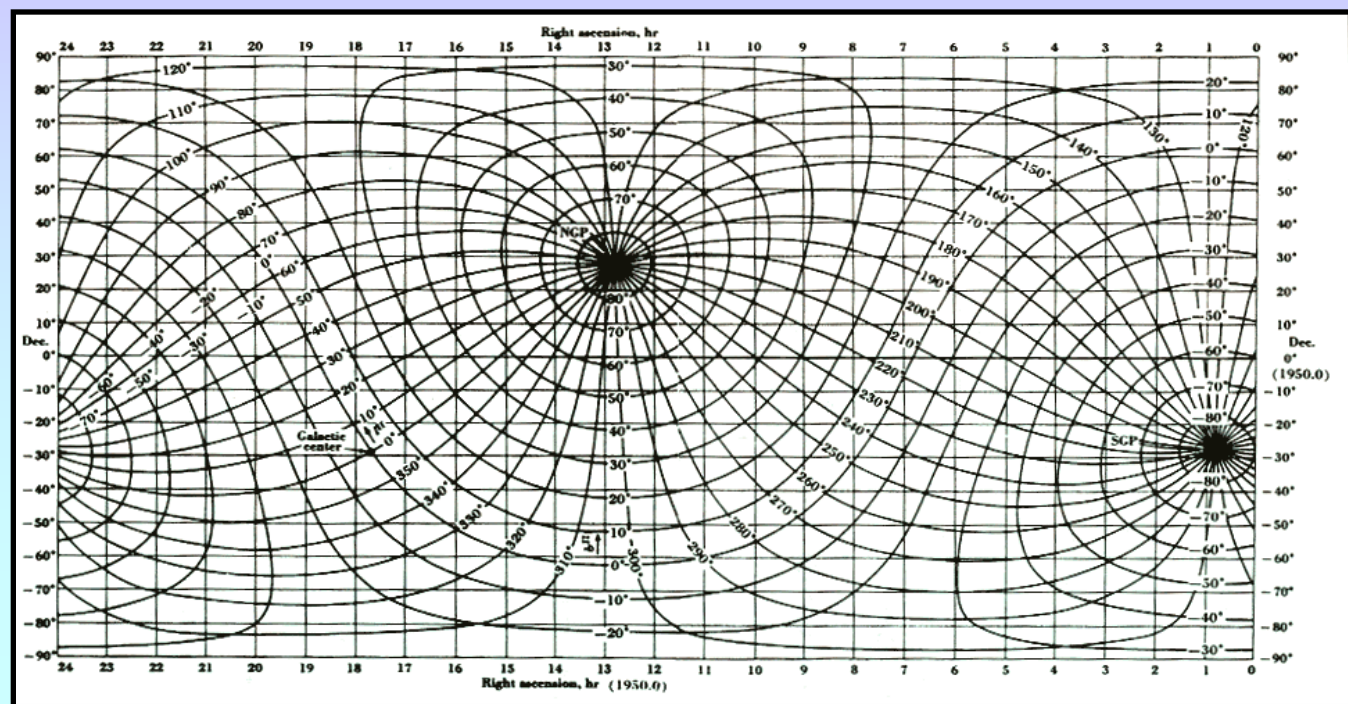
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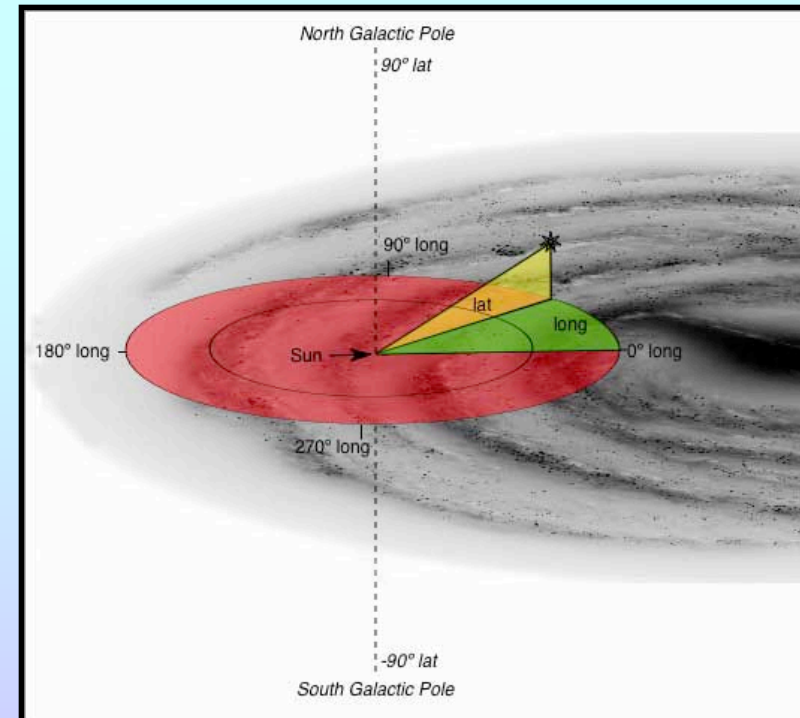


Coordinates in the new Galactic system used to be quoted as ( $\ell^{\text{II}}$ ,  $b^{\text{II}}$ ), but the superscripts have now been dropped.



Like the ecliptic longitude, Galactic longitude is always given in degrees, from  $0^\circ$  to  $360^\circ$ .

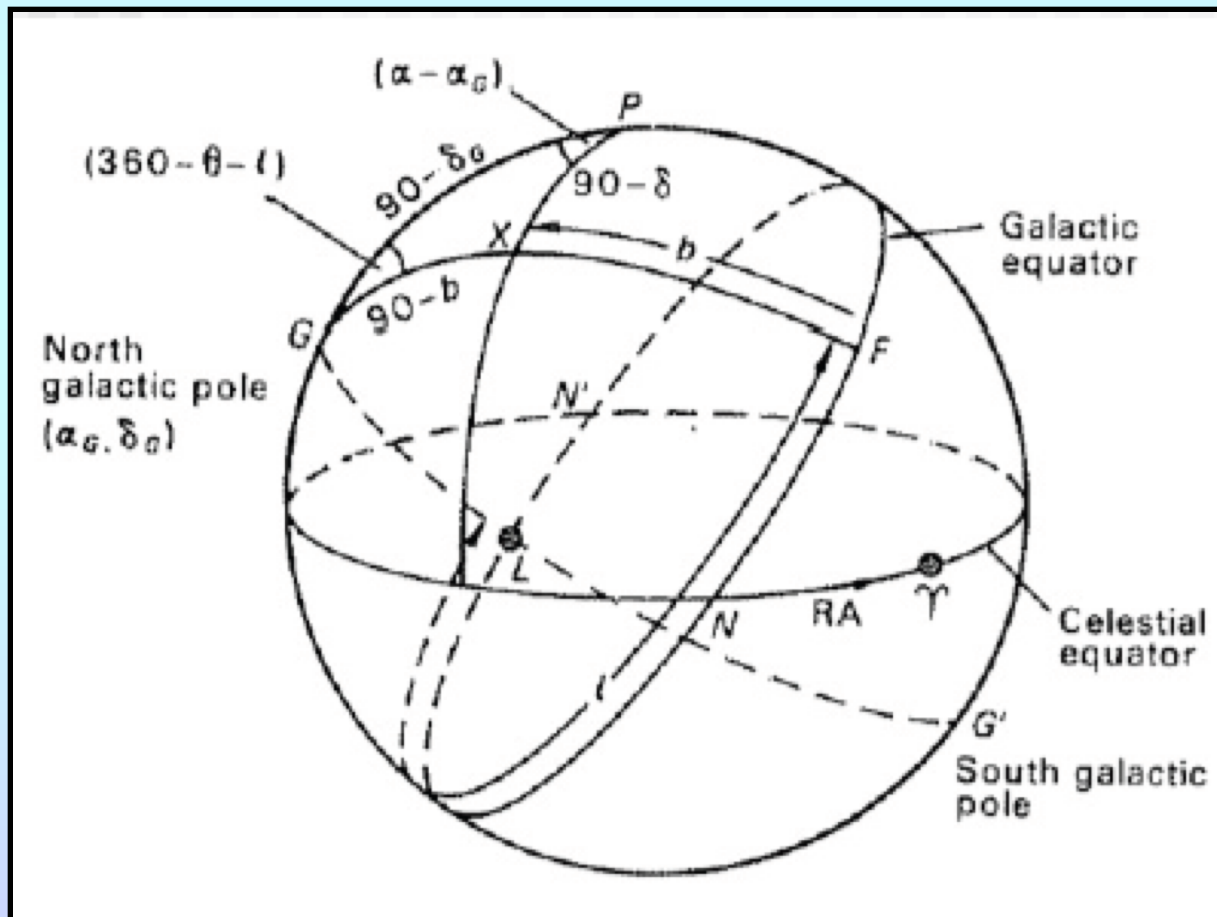
Galactic longitude is now defined such that zero longitude is very close to the Galactic center.





# Equatorial – Galactic Transformations

Transformations between equatorial and Galactic coordinates are more complicated, since they do not share the same zero point for longitude. You can compute them in two ways.



# Equatorial – Galactic Transformations: Spherical Trig

It's possible to derive  $(\ell, b)$  using the fundamental equation of spherical trig, by working in 3 (actually 4) triangles:

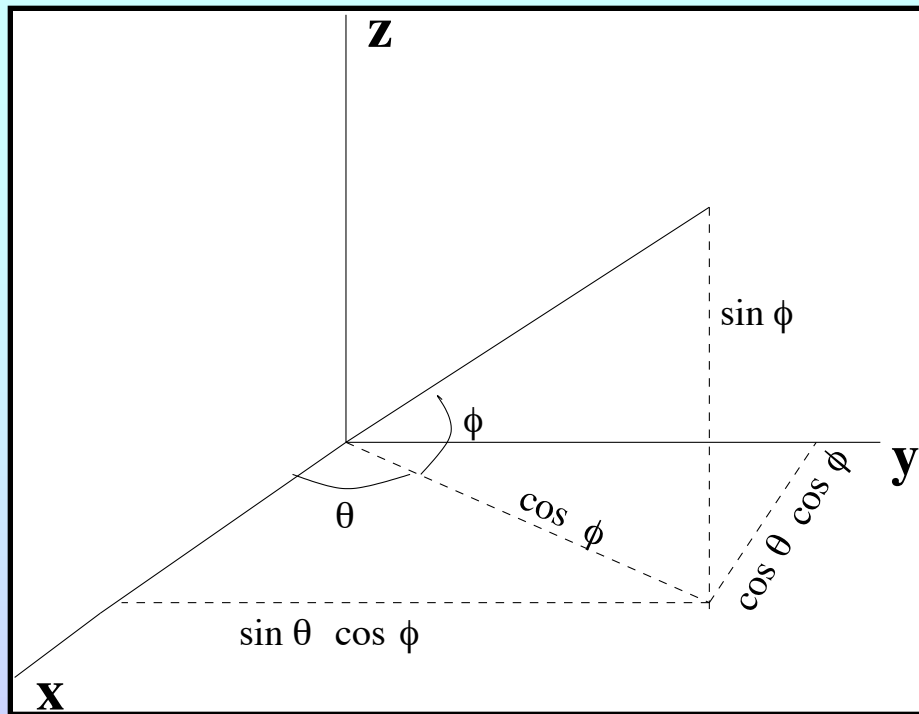
- 1) Galactic latitude can be found from the triangle formed from the object, the North celestial pole, the North Galactic pole.
- 2) The length of an intermediate arc is then computed from a triangle containing the object, the North celestial pole, and the Galactic center.
- 3) Using the intermediate arc, the absolute value for Galactic longitude is found from the triangle that contains the North Galactic Pole, the Galactic Center, and the object.
- 4) To get the longitude sign correct, steps (2) and (3) must then be repeated with the ascending node (the point where the Galactic plane crosses the celestial equator going south to north).



# Equatorial – Galactic Transformations: Euler Angles

A generalized method for converting between coordinate systems is through the use of Euler angles. The procedure is

- Convert to Cartesian coordinates
- Rotate about **z**, **x**, and **z'** axes
- Convert back to spherical coordinates



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \theta \cos \phi \\ \sin \theta \cos \phi \\ \sin \phi \end{pmatrix}$$

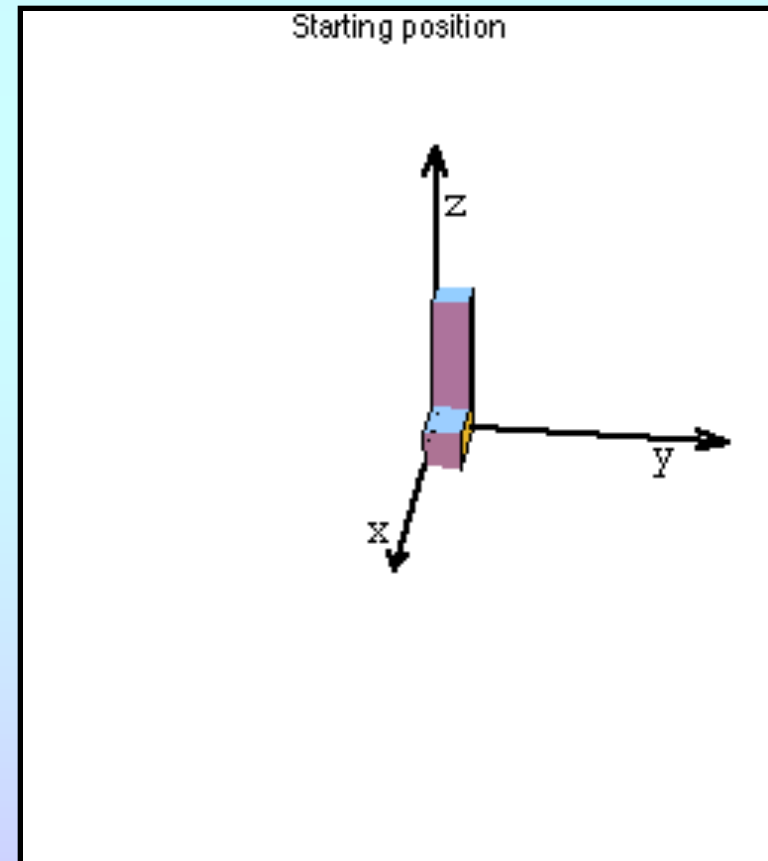
# Equatorial – Galactic Transformations: Euler Angles

A generalized method for converting between coordinate systems is through the use of Euler angles. The procedure is

- Convert to Cartesian coordinates
- Rotate about  $\mathbf{z}$ ,  $\mathbf{x}$ , and  $\mathbf{z}'$  axes
- Convert back to spherical coordinates

Each rotation is a simple matrix multiplication, using 3D rotation matrices. The Euler angles are:

- The angle required to match the ascending node ( $\alpha_0$ )
- The inclination of the Galactic pole from the celestial equator ( $\delta_0$ )
- The angle required to have the Galactic center at  $0^\circ$  ( $\ell_0$ )



# Equatorial - Galactic Transformation

$$\begin{pmatrix} x' \\ y' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \ell_0 & \sin \ell_0 & 0 \\ -\sin \ell_0 & \cos \ell_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \delta_0 & \sin \delta_0 \\ 0 & -\sin \delta_0 & \cos \delta_0 \end{pmatrix} \begin{pmatrix} \cos \alpha_0 & \sin \alpha_0 & 0 \\ -\sin \alpha_0 & \cos \alpha_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

When converted back to spherical coordinates, the relations are

$$\cos b \cos (\ell - \ell_0) = \cos \delta \cos(\alpha - \alpha_0)$$

$$\cos b \sin (\ell - \ell_0) = \cos \delta \sin(\alpha - \alpha_0) \cos \delta_0 + \sin \delta \sin \delta_0$$

$$\sin b = \sin \delta \cos \delta_0 - \cos \delta \sin (\alpha - \alpha_0) \sin \delta_0$$

$$\cos \delta \sin(\alpha - \alpha_0) = \cos b \sin (\ell - \ell_0) \cos \delta_0 - \sin b \sin \delta_0$$

$$\sin \delta = \cos b \sin (\ell - \ell_0) \sin \delta_0 + \sin b \cos \delta_0$$

with

$$\alpha_0(1950) = 282.25^\circ$$

$$\delta_0(1950) = 62.6^\circ$$

$$\ell_0 = 33^\circ$$

# Supergalactic Coordinates

In the 1950's, de Vaucouleurs noticed that most bright galaxies fall in a plane which is roughly perpendicular to the plane of the Milky Way. This is actually the reflection of extragalactic large scale structure. The supergalactic north pole lies at ( $\ell = 47.37^\circ$ ,  $b = +6.32^\circ$ ) and SGB =  $0^\circ$ , SBL =  $0^\circ$  is at ( $\ell = 137.37^\circ$ ,  $b = 0^\circ$ ). Many galaxy surveys use these **Supergalactic coordinates**.

